JOURNAL OF AIRCRAFT
Vol. 42, No. 5, September–October 2005

Engineering Notes

ENGINEERING NOTES are short manuscripts describing new developments or important results of a preliminary nature. These Notes should not exceed 2500 words (where a figure or table counts as 200 words). Following informal review by the Editors, they may be published within a few months of the date of receipt. Style requirements are the same as for regular contributions (see inside back cover).

Equilibrium Manifold Linearization Model for Normal Shock Position Control Systems

Yu Daren,* Cui Tao,† and Bao Wen‡
Harbin Institute of Technology,
150001 Heilongjiang, People's Republic of China

Introduction

A PRIMARY goal of inlet design for supersonic aircraft, such as turbojets and ramjets, is to maintain the normal shock in a stable position just downstream of the throat to minimize total pressure losses and to avoid inlet unstart, which may be caused by very small perturbations.[1,2] Active control of the normal shock position can make possible engine operation close to the stability boundary and, therefore, yield higher inlet pressure recovery.

Complete mathematical description of the dynamic behavior of a normal shock is both complex and highly nonlinear.[3] Linearization is one of the approximate methods by which traditional linear control theories can be applied to analyze and design nonlinear control systems. Hurrell[4] used a linearization method to investigate the effects of downstream pressure disturbances on the normal shock. A linearization model was developed by Willoh[5] to capture the shock motion and by Culick and Rogers[6] and Sajben and Said[7] to analyze the acoustic reflection and transmission properties of a normal shock. More recently, the linearization model was extended by MacMartin[8] to represent the upstream and downstream perturbations as acoustic and entropy waves.

However, all of these models are based on a small-perturbation linearization method and are valid only when the inlet operates about a certain nominal operating position; control systems based on the models must be limited within narrow margins. Therefore, piecewise linear methods are applied to enlarge the operating range. These methods are mathematically characterized by sets of discontinuous small-perturbation models, together with the switching conditions. However, discontinuous switching among different models may decrease the stability of the control system. Therefore, it is of interest to develop a continuous global linearization method.

A nonlinear system may be considered as a family of constant operating points that are defined by an "equilibrium manifold."[9] The basic approach of methods employing an equilibrium manifold is first to approximate a nonlinear plant by linearization about the equilibrium manifold and then to construct a family of linear control laws such that design goals are met at each constant operating point.[10] This Note describes an equilibrium manifold approach for a normal-shock position control system.

Nonlinear Model of Shock Motion

For a stationary normal shock in a flowfield, the steady equations across the normal shock may be written as

$$u_2 = \frac{2a_1^2 + (k-1)u_1^2}{(k+1)u_1} \tag{1}$$

$$a_2 = \frac{\sqrt{\left[2ku_1^2 - (k-1)a_1^2\right]\left[2a_1^2 + (k-1)u_1^2\right]}}{(k+1)u_1} \tag{2}$$

where u, a, and k are the flow velocity, speed of sound, and ratio of specific heats, respectively, and subscripts 1 and 2 denote the locations upstream and downstream of the normal shock.

Assuming that a normal shock satisfies quasi-steady shock equations at each instant of time, we can analyze its unsteady behavior by substituting into the steady shock relations, Eqs. (1) and (2), the instantaneous values of the flow properties.[7] The unsteady shock motion can be expressed as

$$\frac{\mathrm{d}x_s}{\mathrm{d}t} = \frac{[(k-3)u_1 - (k+1)u_2] + \sqrt{16a_1^2 + (k+1)^2(u_1-u_2)^2}}{4} \tag{3}$$

$$\frac{\mathrm{d}x_s}{\mathrm{d}t} = \sqrt{\frac{(k^2-6k+1)a_1^2 + (k+1)^2a_2^2 + \sqrt{16k(k-1)^2a_1^4 + \left[(k+1)^2a_2^2 + (k^2-6k+1)a_1^2\right]^2}}{4k(k-1)}} - u_1 \tag{4}$$

where x_s is the normal shock position.

When it is assumed that conditions upstream are constant in time, the upstream variables u_1 and a_1 can be expressed as functions of x_s, whereas the downstream variables u_2 and a_2 are expressed as the perturbations. Equations (3) and (4) can be written as

$$\frac{\mathrm{d}x_s}{\mathrm{d}t} = f_u(x_s, u_2) \tag{5}$$

$$\frac{\mathrm{d}x_s}{\mathrm{d}t} = f_a(x_s, a_2) \tag{6}$$

where f_u and f_a represent the right-hand sides of Eqs. (3) and (4), respectively.

Equilibrium Manifold Linearization for Modeling of Nonlinear Shock Motion

Based on the equilibrium manifold approach, the key steps of modeling the nonlinear shock motion are introduced in the following section.

Received 15 August 2004; revision received 25 April 2005; accepted for publication 9 May 2005.

*Professor, School of Energy Science and Engineering; yudaren@hcms.hit.edu.cn.
†Ph.D. Student, School of Energy Science and Engineering; cuitao@hcms.hit.edu.cn.
‡Professor, School of Energy Science and Engineering; baowen@hcms.hit.edu.cn.

Equilibrium Manifold Linearization

We begin by considering a multiple-input/multiple-output nonlinear plant described by

$$\dot{x}(t) = f[x(t), u(t)], \qquad y(t) = h[x(t), u(t)] \tag{7}$$

where $f(\cdot)$ and $h(\cdot)$ are smooth functions. The equilibrium manifold of the system described by Eqs. (7) is a set defined by

$$\{(x, u) | f(x, u) = 0\} \tag{8}$$

When it is assumed that the equilibrium manifold can be parameterized by a smooth function $[x(\alpha), u(\alpha), y(\alpha)]$, where α is the scheduling variable, then

$$f[x(\alpha), u(\alpha)] = 0, \qquad h[x(\alpha)] = y(\alpha) \tag{9}$$

Linearizing Eq. (7) about its equilibrium manifold yields the parameterized linearization family[11]

$$\frac{\mathrm{d}}{\mathrm{d}t}[x - x(\alpha)] = A(\alpha)[x - x(\alpha)] + B(\alpha)[u - u(\alpha)]$$

$$y - y(\alpha) = C(\alpha)[x - x(\alpha)] + D(\alpha)[u - u(\alpha)] \tag{10}$$

where

$$A(\alpha) = \frac{\partial}{\partial x} f[x(\alpha), u(\alpha)], \qquad B(\alpha) = \frac{\partial}{\partial u} f[x(\alpha), u(\alpha)]$$

$$C(\alpha) = \frac{\partial}{\partial x} h[x(\alpha), u(\alpha)], \qquad D(\alpha) = \frac{\partial}{\partial u} h[x(\alpha), u(\alpha)] \tag{11}$$

Decoupled Steady State and Dynamic Modeling of Nonlinear Shock Motion

Modeling nonlinear shock motion may be divided into three steps. The first step, steady-state modeling, is to calculate the equilibrium manifold; the second step, dynamic modeling, is to calculate the partial differential coefficients; and finally, combined modeling, is to simulate the nonlinear shock motion.

Steady-State Modeling

Under the assumption that conditions upstream are constant, we can define the equilibrium manifold of the nonlinear shock motion as

$$[x_s(\alpha), u_2(\alpha)] | f_u[x_s(\alpha), u_2(\alpha)] = 0$$

$$[x_s(\alpha), a_2(\alpha)] | f_a[x_s(\alpha), a_2(\alpha)] = 0 \tag{12}$$

Dynamic Modeling

A normal shock can be thought as a discontinuous interface that divides the airflow into two sections: high-density and low-density. The motion of the interface changes the mass, momentum, and energy of the flow passing through it. When the normal shock is finally stable in a new position, the flow is brought into a new balance. In the process, the perturbation energy is partially dissipated by shock compression and viscosity, and the rest of the energy is deposited into the flow passing through the normal shock. Therefore, it is evident that the fundamental mechanism of the dynamics of shock motion is the energy storage effects of the flow passing through a shock.

From Eqs. (5), (6), (10), and (11), the partial differential coefficients are calculated as

$$A_u(\alpha) = \frac{\partial}{\partial x_s} f_u[x_s(\alpha), u_2(\alpha)] = \frac{u_1\left[(k-1)M_1^2 + 2\right]}{2\left(1 - M_1^4\right)} \frac{1}{A} \frac{\mathrm{d}A}{\mathrm{d}x} \tag{13}$$

$$B_u(\alpha) = \frac{\partial}{\partial u_2} f_u[x_s(\alpha), u_2(\alpha)] = -\frac{1}{4}\left[(k+1) + \frac{(k+1)^2(u_1 - u_2)}{\sqrt{16a_1^2 + (k+1)^2(u_1 - u_2)^2}}\right] \tag{14}$$

$$A_a(\alpha) = \frac{\partial}{\partial x_s} f_a[x_s(\alpha), a_2(\alpha)] = \frac{u_1\left[(k-1)M_1^2 + 2\right]^2}{4\left(1 - M_1^2\right)\left(kM_1^4 + 1\right)} \frac{1}{A} \frac{\mathrm{d}A}{\mathrm{d}x} \tag{15}$$

$$B_a(\alpha) = \frac{\partial}{\partial a_2} f_a[x_s(\alpha), a_2(\alpha)]$$

$$= \frac{a_2\left\{(k+1)^2 + (k+1)^2\left[a_2^2(k+1)^2 + a_1^2(k^2 - 6k + 1)\right] \Big/ \sqrt{16k(k-1)^2 a_1^4 + \left[(k+1)^2 a_2^2 + (k^2 - 6k + 1)a_1^2\right]^2}\right\}}{4\sqrt{k(k-1)}\sqrt{(k^2 - 6k + 1)a_1^2 + (k+1)^2 a_2^2 + \sqrt{16k(k-1)^2 a_1^4 + \left[(k+1)^2 a_2^2 + (k^2 - 6k + 1)a_1^2\right]^2}}} \tag{16}$$

(17)

where A and M are the cross-sectional area and Mach number, respectively.

Combined Modeling

When the equilibrium manifold and the partial differential coefficients are calculated, we obtain the final equations as

$$\frac{\mathrm{d}}{\mathrm{d}t}[x_s - x_s(\alpha)] = A_u(\alpha)[x_s - x_s(\alpha)] + B_u(\alpha)[u_2 - u_2(\alpha)] \tag{17}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[x_s - x_s(\alpha)] = A_a(\alpha)[x_s - x_s(\alpha)] + B_a(\alpha)[a_2 - a_2(\alpha)] \tag{18}$$

Choice of the Scheduling Variable

The scheduling variable determines which constant operating point the control system is automatically operated about. Therefore, the choice of the scheduling variable is a critical step of the method. As discussed in Ref. 12, the scheduling variable should have relatively slow motion compared to the scheduled variables, because the equilibrium manifold linearization method does not provide explicit guarantees about behaviors of the system when it is necessary to transit from one operating point to another at a very high speed.

Compared to other variables, the movement of x_s is to some degree continuous and slow because of the inertia of shock motion. Therefore, we define x_s as the scheduling variable,

$$\alpha = x_s \tag{19}$$

Error Estimation

The equilibrium manifold linearization is based on Taylor series expansion to the first order about its equilibrium manifold subspace, and the quality of equilibrium manifold linearization mainly depends on the properties of the second-order Taylor series expansion about the subspace.

From Eqs. (17) and (18), we obtain the model errors as

$$E_u(x_s, u_2) = \frac{1}{2}\frac{\partial^2 f_u(x_s, \xi_u)}{\partial x_s^2}[x_s - x_s(x_s)]^2 + \frac{1}{2}\frac{\partial^2 f_u(x_s, \xi_u)}{\partial u_2^2}[u_2 - u_2(x_s)]^2 = \frac{1}{2}\frac{\partial^2 f_u(x_s, \xi_u)}{\partial u_2^2}[u_2 - u_2(x_s)]^2 \tag{20}$$

$$E_a(x_s, a_2) = \frac{1}{2}\frac{\partial^2 f_a(x_s, \xi_a)}{\partial a_2^2}[x_s - x_s(x_s)]^2 + \frac{1}{2}\frac{\partial^2 f_a(x_s, \xi_a)}{\partial a_2^2}[a_2 - a_2(x_s)]^2 = \frac{1}{2}\frac{\partial^2 f_a(x_s, \xi_a)}{\partial a_2^2}[a_2 - a_2(x_s)]^2 \tag{21}$$

where the value of ξ_u is between u_2 and $u_2(x_s)$ and that of ξ_a is between a_2 and $a_2(x_s)$.

It can be seen that the model errors are determined by the second-order Taylor series expansion of the nonlinear functions f_u and f_a and the distance from the operating point to its equilibrium manifold. When the system operates within the neighborhood of the equilibrium manifold, the error is mainly determined by the second-order Taylor series expansion coefficients $\partial^2 f_u(x_s, \xi_u)/\partial u_2^2$ and $\partial^2 f_a(x_s, \xi_a)/\partial a_2^2$.

Simulation Results

Simulations were done to demonstrate the validity of the equilibrium manifold linearization model for a normal shock position control system.

First we evaluate the accuracy of the equilibrium manifold linearization model by the error formulas described by Eqs. (20) and (21). The first and second Taylor series expansion coefficients of the nonlinear functions f_u and f_a are shown in Figs. 1 and 2. The range of $\partial f_u(x_{se}, u_{2e})/\partial u_{2e}$ is from -0.68 to -0.89, and that of $\partial f_u(x_{se}, a_{2e})/\partial a_{2e}$ is from 3.34 to 3.47, whereas the range of $\partial f_u^2(x_{se}, u_{2e})/\partial u_{2e}^2$ is from 5.0×10^{-4} to 7.0×10^{-4}, and that of $\partial f_a^2(x_{se}, a_{2e})/\partial a_{2e}^2$ is from -3.92×10^{-3} to 8.61×10^{-3}. It is clear that the second Taylor series expansion coefficients are much smaller than the first Taylor series expansion coefficients.

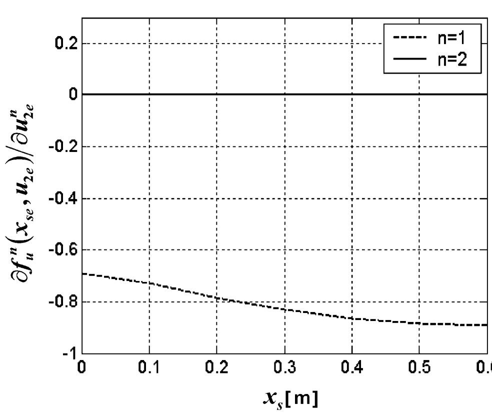

Fig. 1 Taylor series expansion coefficients $\partial f_u^n(x_{se}, u_{2e})/\partial u_{2e}^n$.

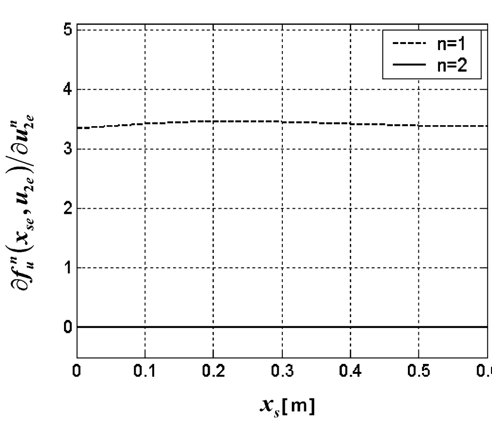

Fig. 2 Taylor series expansion coefficients $\partial f_a^n(x_{se}, a_{2e})/\partial a_{2e}^n$.

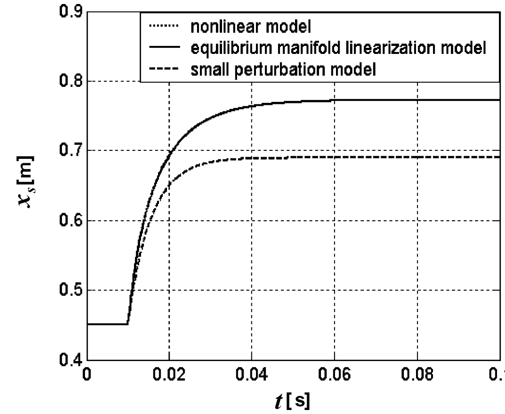

Fig. 3 Shock position response to step change of u_2.

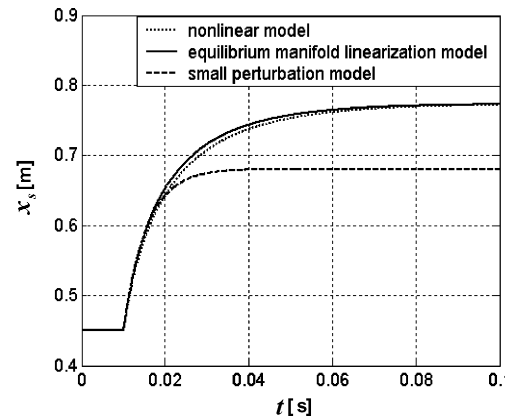

Fig. 4 Shock position response to step change of a_2.

Next we compare the simulation results from the equilibrium manifold linearization model described by Eqs. (17) and (18), the nonlinear model described by Eqs. (5) and (6), and the small-perturbation model of Ref. 8. Figure 3 shows the normal shock position responses to a step change from 435 to 360 m/s of the perturbation variable u_2, where the incoming flow Mach number is 2.5 and the ratio of the cross-sectional areas is $\frac{1}{2}$. The maximum dynamic error of -2.2×10^{-3} m and steady-state error of -1.8×10^{-6} m demonstrate that the equilibrium manifold linearization model is of high accuracy. The same observation is made in Fig. 4, in which maximum dynamic error is 9.5×10^{-3} m and steady-state error is 8.3×10^{-5} m for a step change from 532.6 to 543.7 m/s of the perturbation variable a_2. It is also seen in Figs. 3 and 4 that the accuracy of the equilibrium manifold linearization model is much higher than that of the small-perturbation model of Ref. 8.

Conclusions

A new method for modeling nonlinear shock motion based on the equilibrium manifold linearization method is proposed. The method approximates the nonlinear system of shock motion by a linearization family, which is more accurate than the small-perturbation linearization method and is simpler than the piecewise linear method. The equilibrium manifold approach provides an improvement over these methods. Error estimation and simulation results show its validity.

References

[1]Susumu, M., and Keiichi, K., "Unstart Phenomenon Due to Thermal Choke in Scramjet Module," *10th AIAA/NAL–NASDA–ISAS International Space Planes and Hypersonic Systems and Technologies Conference*, AIAA, Reston, VA, 2001, pp. 1–7.

[2]Mayer, D. W., and Paynter, G. C., "Prediction of Supersonic Inlet Unstart Caused by Freestream Disturbances," *AIAA Journal*, Vol. 33, No. 2, 1995, pp. 266–275.

[3]Salerno, M., and Malomed, B. A., "Shock Wave Dynamics in a Discrete Nonlinear Schrodinger Equation with Internal Losses," *Physical Review E: Statistical Physics, Plasmas, Fluids, and Related Interdisciplinary Topics*, Vol. 62, No. 6, 2000, pp. 8651–8656.

[4]Hurrell, H. G., "Analysis of Shock Motion in Ducts During Disturbances in Downstream Pressure," NACA TN-4090, Sept. 1957.

[5]Willoh, R. G., "A Mathematics Analysis of Supersonic Inlet Dynamic," NASA TND-4969, Aug. 1968.

[6]Culick, F. E. C., and Rogers, T., "The Response of Normal Shocks in Diffusers," *AIAA Journal*, Vol. 21, No. 10, 1983, pp. 1382–1390.

[7]Sajben, M., and Said, H., "Acoustic-Wave/Blade-Row Interactions Establish Boundary Conditions for Unsteady Inlet Flows," *Journal of Propulsion and Power*, Vol. 17, No. 5, 2001, pp. 1090–1099.

[8]MacMartin, D. G., "Dynamics and Control of Shock Motion in a Near-Isentropic Inlet," *Journal of Aircraft*, Vol. 41, No. 4, 2004, pp. 846–853.

[9]Shin, M.-H., and Chung, M. J., "Parameterization and Stabilization of the Equilibrium Points of Affine Non-linear Control Systems," *International Journal of Systems Science*, Vol. 33, No. 4, 2002, pp. 301–311.

[10]Lawrence, D. A., "Approximate Model Matching for Nonlinear Systems," *Proceedings of the 38th IEEE Conference on Decision and Control*, IEEE Press, Piscataway, NJ, Dec. 1999, pp. 720–725.

[11]Harry, G. K., and Chang, B. C., "Constructing Linear Families from Parameter-Dependent Nonlinear Dynamics," *IEEE Transactions on Automatic Control*, Vol. 43, No. 8, 1998, pp. 1143–1147.

[12]Sandberg, I. W., "*Lp* Stability and Linearization," *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, Vol. 49, No. 10, 2002, pp. 1437–1441.